✱ Correction: [Wicher MALTEN] X CW-complex ⇒ ΩX has homotopy type of CW-complex [Milnor]

So $\Omega K(G,n) \underset{he.}{\simeq} K(G,n-1)$

Vector bundles:

Def: $\pi : E \to B$ vector bundle (of rank r) over a base space B := fiber bundle in which each fiber $E_b = \pi^{-1}(b)$ is equipped w/ a structure of vector space (over say $\mathbb{R}$ or $\mathbb{C}$), s.t. $\exists$ local trivializations

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\sim", "h_i"'] \arrow[dr] & & U_i \times \mathbb{R}^r \arrow[dl] \\ & U_i & \end{tikzcd}$$

h_i homeomorphism, linear isom. on each fiber.

Ex:
- trivial bundle $B \times \mathbb{R}^r$
- tangent bundle of a smooth manifold $\begin{matrix} TM \\ \pi \downarrow \\ M \end{matrix}$

  (for $M \subseteq \mathbb{R}^N$, $TM = \{(x,v) \in M \times \mathbb{R}^N \,/\, v \text{ tangent to } M \text{ at } x\}$)

(Recall: smooth manifold M = $\exists$ open cover by coordinate charts $U_i \xrightarrow{\varphi_i} \mathbb{R}^n$, with transition functions (coord. changes) $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_j(U_i \cap U_j) \xrightarrow{\sim} \varphi_i(U_i \cap U_j)$ C^∞ diffeomorphisms

This gives a notion of smooth functions (= smooth in local coords)

In local chart,

$$\begin{tikzcd} TM_{|U_i} \arrow[r, "\sim", "D\varphi_i"'] \arrow[d] & \varphi_i(U_i) \times \mathbb{R}^n \arrow[d] \\ U_i \arrow[r, "\sim", "\varphi_i"'] & \varphi_i(U_i) \end{tikzcd}$$

with change of coords $(x,v) \mapsto (\varphi_{ij}(x), (D\varphi_{ij})_x(v))$

- normal bundle to a submanifold $M \subseteq \mathbb{R}^n$

  $$NM = \{(x,v) \in M \times \mathbb{R}^n \,/\, v \perp T_x M\}$$

  or more generally to a submanifold inside a Riemannian manifold X (see later)

- tautological bundle over $\mathbb{RP}^n$: real line bundle (= rank 1)

  $$E = \{(x,\xi) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \,/\, \xi \in \text{line corresp. to } x\}$$

  (i.e. thinking of x as a point of S^n up to ± 1, ask $\exists t \in \mathbb{R} / \xi = tx$)

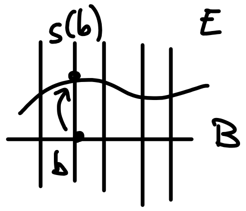
Similarly, tautological k-plane bundle over $Gr_k(\mathbb{R}^n)$

$$E = \{(V, \xi) \,/\, V \in Gr_k(\mathbb{R}^n) \text{ } k\text{-plane} \subset \mathbb{R}^n, \quad \xi \in V\}$$

and similarly over $\mathbb{CP}^n$, $Gr_k(\mathbb{C}^n)$

All these are in fact smooth vector bundles over smooth manifolds. (i.e. everything C^∞).

Def: A section of a vector bundle $\pi : E \to B$ is a map $s : B \to E$ st. $\pi \circ s = id$.



(resp. smooth section of a smooth vector bundle).

Ex: vector field on a mfld M $\longleftrightarrow$ section of TM.

(trivial := isom. to product bundle; $E, E' \to B$ isom. if $\exists$
$$\begin{tikzcd} E \arrow[rr, "\sim"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$
fiberwise linear iso.)

Prop: A rank r vector bundle is trivial iff $\exists$ r sections $s_1, \dots, s_r$ which are pointwise linearly independent.

Pf. If trivial then $E \simeq B \times \mathbb{R}^r$, take sections $s_i(b) = (b, (0 \dots 0 \underset{\overset{\uparrow}{i}}{1} 0 \dots 0))$

conversely, such sections s_i define a trivialization
$$E \ni (b, \underbrace{\sum_{i=1}^{r} t_i \, s_i(b)}_{\text{any elt of } E_b \text{ is of this form}}) \longleftrightarrow (b, (t_1 \dots t_r)) \in B \times \mathbb{R}^r$$

∎

Ex: The tautological line bundle $E \xrightarrow{\pi} \mathbb{R}P^n$ is not trivial.

Indeed, otherwise $\exists$ nowhere vanishing section $s : \mathbb{R}P^n \to E$
i.e. $\forall x \in \mathbb{R}P^n$, $s(x) \in \mathbb{R}^{n+1} \setminus \{0\}$ with $\pi(s(x)) = x$. Replace $s(x)$ by $\frac{s(x)}{|s(x)|}$ to get a section
$$\begin{tikzcd} s : \mathbb{R}P^n \arrow[r, shift left, "s"] & S^n \arrow[l, shift left, "\pi"] \end{tikzcd} \quad \pi \circ s = id$$
(π univ cover).

Contradiction (implies covering is trivial).

Euclidean vector bundles:

Def: An Euclidean vect bundle = real vect. bundle $E \xrightarrow{\pi} B$ + Euclidean metric: continuous (/smooth) function $h = |\cdot|^2 : E \to \mathbb{R}$ whose restriction to each fiber is a definite positive quadratic form (hence induces a Eucl. scalar product).

Ex: Eucl. metric on $TM \to M$ $\underset{\text{def.}}{\Longleftrightarrow}$ Riemannian metric on M

Lemma: Every vector bundle over a paracompact (eg metric) base B admits a Euclidean metric

(Exercise; use partition of unity)

Lemma: A rank r Euclidean vector bundle is trivial iff it admits r pointwise orthonormal sections $s_1, \dots, s_r$.

(Pf: start with linearly indep. sections & apply Gram-Schmidt orthonormalization. This preserves continuity: $s/|s|$, $s - \langle s, s' \rangle s'$ ✓).

Constructions :

- **pullback:** $f : A \to B$ with $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ $\Rightarrow f^*E = \{(a,v) \in A \times E \,/\, f(a) = \pi(v)\}$ vector bundle over A,

$$\begin{tikzcd} f^*E \arrow[r, "\hat{f}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Ex: pullback by inclusion $A \subset B$ is the restriction $E_{|A}$.

- **bundle maps:** different notions

(a) $f : E_1 \to E_2$ st.

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

$\exists\, g : B_1 \to B_2$ making diagram commute + on each fiber, $f_b : (E_1)_b \to (E_2)_{g(b)}$ linear iso.

Then $E_1 \simeq g^*E_2$.

(b) relax by requiring f_b linear maps of constant rank indep of b.
(then $\mathrm{Ker}\, f$, $\mathrm{Im}\, f$ are subbundles of E_1, E_2)

(c) require $g = id$, ie. $f : E_1 \to E_2$ linear maps of const. rank.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr] & & E_2 \arrow[dl] \\ & B & \end{tikzcd}$$

- **Whitney sum:** $E_1, E_2 \xrightarrow{\pi_i} B \Rightarrow E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \,/\, \pi_1(v_1) = \pi_2(v_2)\}$
has fiber $(E_1 \oplus E_2)_b \cong (E_1)_b \oplus (E_2)_b$.

Can construct as pullback

$$\begin{tikzcd} E_1 \oplus E_2 \arrow[r] \arrow[d] & E_1 \times E_2 \arrow[d, "\pi_1 \times \pi_2"] \\ B \arrow[r, "\delta"] & B \times B \end{tikzcd}$$

(obviously a vect bundle)

$x \longmapsto (x,x)$ diagonal

Easy: $F_1, F_2 \subset E$ subbundles with $E_b = (F_1)_b \oplus (F_2)_b \ \forall b \in B \Rightarrow E \underset{\text{isom.}}{\simeq} F_1 \oplus F_2$.

given $F \subset E$ subbundle, $\exists ?\ F'$ subbundle st. $E \underset{\text{isom.}}{\simeq} F \oplus F'$?

$\Rightarrow$ • **Orthogonal complement:**

Prop: $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ Euclidean vect bundle, $F \subset E$ subbundle $\Rightarrow$

$F^\perp = \{v \in E \,/\, \langle v, w \rangle = 0 \ \forall w \in F_{\pi(v)}\}$ is a subbundle, and $E = F \oplus F^\perp$

(Pf. is essentially Gram-Schmidt again: start w/ local trivs. of E & F to pick local orthonormal frames of sections $s_1 \dots s_r$ of $F_{|U}$, $\sigma_1 \dots \sigma_n$ of $E_{|U}$ ($b \in U \subset B$ open)
Expressing $s_1 \dots s_r$ in basis $\sigma_1 \cdots \sigma_n$ gives map $U \to \mathrm{Mat}_{r \times n}$, of full rank.

Can assume first r columns lin indep^t at given $b \in U$ by relabelling σ_i's.

By continuity, $\exists \underset{\text{open}}{V} \ni b$ over which this remains true.

Then Gram-Schmidt applied to $(s_1 \dots s_r, \sigma_{r+1}, \dots \sigma_n) \rightsquigarrow (\underbrace{s_1 \dots s_r}_{\text{o.n.b. } F}, \underbrace{t_{r+1} \dots t_n}_{\text{o.n.b. } F^\perp})$

gives loc. triv. of $F^\perp$ ▲

Ex: $Q \subset M$ submanifold, M Riemannian metric $\Rightarrow$ TQ subbundle of $TM|_Q$

$\rightsquigarrow$ normal bundle $NQ = TQ^\perp \subset TM|_Q$, so $TM|_Q = TQ \oplus NQ$

In fact, same can be done for an immersion $f: Q \to M$, TQ is a subbundle of $f^* TM$

- Continuous functorial operations:

$E, F \to B \rightsquigarrow$ $E \oplus F$ already seen
$\mathrm{Hom}(E,F)$ with fibre $\mathrm{Hom}(E,F)_b = \mathrm{Hom}(E_b, F_b)$
$E \otimes F$
$E^* = \mathrm{Hom}(E, \underline{\mathbb{R}})$
$\Lambda^k E$ exterior powers ....

point: there are functorial operations on vector spaces, ie

ie. $V_1 \xrightarrow[\sim]{f} V_2$, $W_1 \xrightarrow[\sim]{g} W_2$ induce $V_1 \otimes W_1 \xrightarrow[\sim]{f \otimes g} V_2 \otimes W_2$
continuous, ie. depends C^0 on f, g.

So thinking of $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ as $\bigcup_\alpha (U_\alpha \times \mathbb{R}^r) \big/ (x \in U_\alpha, v) \sim (x \in U_\beta, \varphi^E_{\alpha\beta}(x) v)$

$\varphi^E_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow[C^0]{} GL(r)$

then $E \otimes F$ has gluings $\varphi^E_{\alpha\beta} \otimes \varphi^F_{\alpha\beta}$ ▲

Ex: The normal bundle can also be built w/out using a Riem. metric as quotient by subbundle, $NQ = TM|_Q / TQ$

HW: 2C ($\exists$ Eucl. metrics)
3B/3C (quotient bundle, ker/coker).

Next: Stiefel-Whitney classes.